Trigonometric fourier series

We have, synthesis eq

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

This expansion is complex ~~form~~ exponential ~~form~~ fourier series.

$$= \cdots + a_{-5} e^{-j5\omega_0 t} + a_{-4} e^{-j4\omega_0 t} + a_{-3} e^{-j3\omega_0 t} + a_{-2} e^{-j2\omega_0 t} + a_{-1} e^{-j\omega_0 t} + a_0$$
$$+ a_1 e^{j\omega_0 t} + a_2 e^{j2\omega_0 t} + a_3 e^{j3\omega_0 t} + a_4 e^{j4\omega_0 t} + a_5 e^{j5\omega_0 t} + \cdots$$

let

$$a_1 e^{j\omega_0 t} + a_{-1} e^{-j\omega_0 t}$$

Case I :- $a_1 = a_{-1} = 1$

$$\Rightarrow e^{j\omega_0 t} + e^{-j\omega_0 t} = \left(\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right) \times 2 = 2\cos\omega_0 t$$

Case II :- $a_1 = 1, \; a_{-1} = -1$

$$\Rightarrow e^{j\omega_0 t} - e^{-j\omega_0 t} = \left(\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}\right) \times 2j = 2j\sin\omega_0 t$$

Case III :- $a_1 = 1+2j, \; a_{-1} = 1-2j$

$$\Rightarrow (1+2j) e^{j\omega_0 t} + (1-2j) e^{-j\omega_0 t}$$

$$= 1.\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right) + 2j\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right)$$

$$= 2 \times \left(\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right) + 2j\left(\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}\right) \times 2j$$

$$= 2\cos\omega_0 t + 2j\sin\omega_0 t$$

similarly, for $a_2, a_3, a_4 \ldots$ & $a_{-2}, a_{-3}, \ldots a_{-k}$
a_k

After final Calculation:

$x(t) = a_0 +$ Cosine terms $+$ Sine terms

$$\boxed{x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t}$$ – Trigonometric fourier series.

Note:- Both series are same but representation in different form

Synthesis eqn expression:.

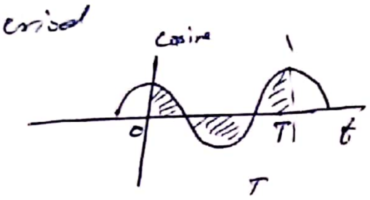
(1) $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$ $\longrightarrow$ Complex exponential F.S.

(2) $x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$ $\rightarrow$ Trig. F.S.

Let us solve (2)

$$x(t) = a_0 + a_1 \cos\omega_0 t + a_2 \cos 2\omega_0 t + a_3 \cos 3\omega_0 t + \cdots a_n \cos n\omega_0 t$$
$$+ b_1 \sin\omega_0 t + b_2 \sin 2\omega_0 t + b_3 \sin 3\omega_0 t + \cdots b_n \sin n\omega_0 t \quad \text{—(1)}$$

Integrating both sides w.r.t for one Complete period



$$\int_0^T x(t)\,dt = \int_0^T a_0\,dt + 0$$

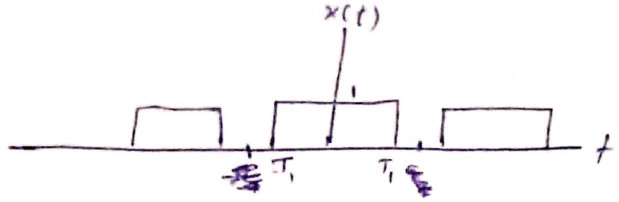
$$\int_0^T \sin\omega_0 t\,dt = 0 \ \& \int_0^T \cos\omega_0 t\,dt = 0$$

$$\int_0^T \sin n\omega_0 t \cos n\omega_0 t\,dt = 0$$

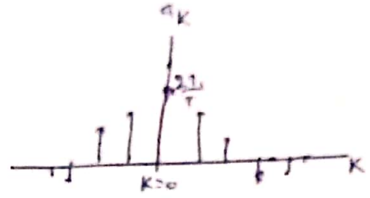
$$\int_0^T x(t)\,dt = a_0 \int_0^T dt = a_0 \cdot T$$

$$\boxed{a_0 = \frac{1}{T}\int_0^T x(t)\,dt}$$ $\rightarrow$ It is an average power/average value of the signal.

We have seen for gate pulse



$\Rightarrow$



We solve previously

$$a_K = \frac{\sin K\omega_0 T_1}{\pi K}$$

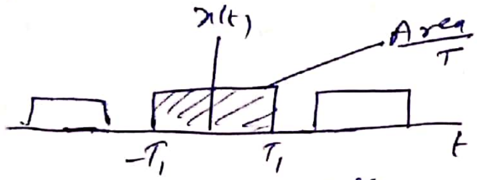
We put $K=0$, $a_0 = \frac{0}{0}$

we done L-hospital rule & find

$$a_0 = \frac{2T_1}{T}$$

But know, we can directly find a_0

By finding average value of above signal.



$$\frac{\text{Area}}{T} = \frac{1 \times 2T_1}{T} = a_0$$

Area

$$a_0 = \frac{1}{T}\int_0^T x(t)$$

Let us solve (2), From eq (1)

multiplying eq (1), both sides with $\cos n\omega_0 t$ & integrating w.r.t time for 0 to T.

$$\int_0^T x(t)\cos n\omega_0 t\cdot dt = \underbrace{\int_0^T a_0 \cos n\omega_0 t\, dt}_{0} + \underbrace{\int_0^T a_1 \cos\omega_0 t\cdot\cos n\omega_0 t\, dt}_{0}$$

$$+ \cdots + \int_0^T a_n \cos n\omega_0 t\cdot\cos n\omega_0 t\, dt + \underbrace{\int_0^T b_1 \sin\omega_0 t \cos n\omega_0 t\, dt}_{0}$$

$$+ \underbrace{\int_0^T b_2 \sin 2\omega_0 t \cos n\omega_0 t\, dt}_{0} + \cdots \underbrace{\int_0^T b_n \sin n\omega_0 t \cos n\omega_0 t\, dt}_{0}$$

We know

$$\int_0^T \sin n\omega_0 t\, dt = 0$$

$$\int_0^T \cos\omega_0 t\, dt = 0$$

$$\int_0^T \sin n\omega_0 t\cdot\cos n\omega_0 t\, dt = 0$$

$$\int_0^T \cos x_1 t\cdot\cos x_2 t\, dt = 0$$

$$\int_0^T x(t)\cos n\omega_0 t\, dt = a_n \int_0^T \cos^2 n\omega_0 t\, dt = a_n \int_0^T \left[\frac{1+\cos 2n\omega_0 t}{2}\right] dt$$

$$= a_n \int_0^T \frac{1}{2}\, dt + a_n \underbrace{\int_0^T \frac{\cos 2n\omega_0 t}{2}\, dt}_{0}$$

$$\int_0^T x(t)\cos n\omega_0 t\, dt = \frac{a_n}{2}\cdot T$$

$$\boxed{a_n = \frac{2}{T}\int_0^T x(t)\cos n\omega_0 t\, dt}$$

Similarly,

$$\boxed{b_n = \frac{2}{T}\int_0^T x(t)\sin n\omega_0 t\, dt}$$

Application of Fourier Series :-

If $x(t)$ is periodic signal

$$x(t) \xleftrightarrow{F.S.} a_K$$

& Fourier series representation,

$$x(t) = \sum_{k=-\infty}^{\infty} a_K e^{jk\omega_0 t}$$

if this signal $x(t)$ is given as input to a C-T LTI system with impulse response $h(t)$,

then,

$$y(t) = \sum_{k=-\infty}^{\infty} a_K H(jk\omega_0) e^{jk\omega_0 t}$$

where

$a_K \rightarrow$ F.S. coeff of $x(t)$

$H(jk\omega_0) \rightarrow$ freq. response of the system

$y(t) \rightarrow$ is also periodic with the same fundn period as $x(t)$

Note:- $a_K \rightarrow$ set of Fourier series coeff ~~of x(t)~~ for input $x(t)$

$a_K H(jk\omega_0) \rightarrow$ set of F.S coeff ~~of y(t)~~ for output $y(t)$

Convergence of Fourier series

A periodic signal $x(t)$ has a Fourier series representation if it satisfy the following Dirichlet conditions:-

Let $x(t)$ be a single-valued periodic function. Then,

1) $x(t)$ has at most a finite number of discontinuities in one period.
2) $x(t)$ has at most a finite number of maxima and minima in one period.
3) $x(t)$ is bounded. That is

$$\int |x(t)|\, dt < \infty$$

Note:- These conditions are sufficient but not necessary conditions for the Fourier series representation.

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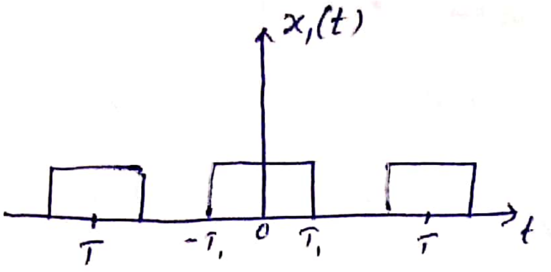
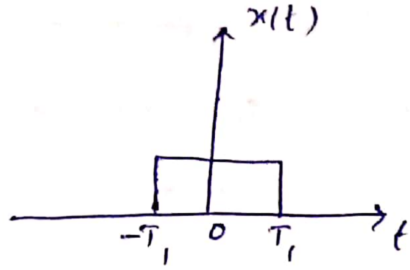
Fourier Representation for Aperiodic signals

(Fourier Transform)



(a) C-T periodic signal



(b) C-T aperiodic signal.

As we know, A periodic signal $x(t)$ with period T and its exponential fourier series coeff a_K or $X(K)$ are related by

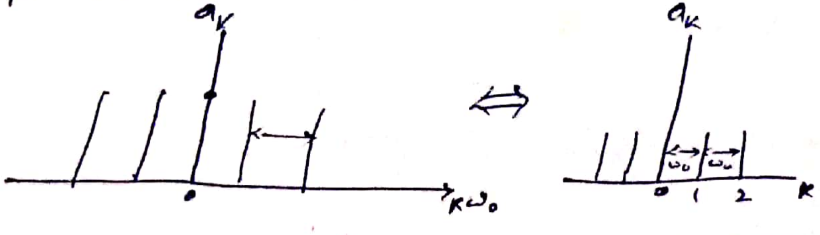
$$x(t) = \sum_{K=-\infty}^{\infty} a_K e^{jK\omega_0 t} \quad \text{——} \ (1)$$

$$a_K = \frac{1}{T}\int_0^T x(t)\, e^{-jK\omega_0 t}\, dt = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jK\omega_0 t}\, dt \quad \text{——} \ (2)$$

→ If the period T is ~~to~~ stretched without limit, the periodic signal no longer remains periodic but becomes a single pulse $x(t)$, corresponding to one period of the periodic pulse, $x(t)$.

→ The change over from periodic to aperiodic signal also represents a transition from a power signal to an energy signal.

we know the FS spectrum for a_K



→ As $T \to \infty$, $\omega_0 = \frac{2\pi}{T} \to 0$ for aperiodic signal.

→ The line spectrum becomes a continuous spectrum

→ Also, If we replace ω_0 by a infinitesimally small quantity $d\omega \to 0$, the discrete freq '$k\omega_0$' may be replace by a continuous freq 'ω'.

→ From eq ②, ~~if we eliminate the~~ The factor $\frac{1}{T}$ means that $a_k \to 0$ as $T \to \infty$. If we eliminate the dependence of ~~x(t)~~ a_k on $\frac{1}{T}$ in the integral and work with Ta_k, we get.

$$T a_k = \int_{-T/2}^{T/2} x(t)\, e^{-jk\omega_0 t}\, dt \quad — ③$$

The integral on the RHS of the eq ③ often exists as $T \to \infty$ (even though Ta_k is in indeterminate form), & we obtain meaningful results.

Further, because $k\omega_0 \to \omega$, the integral describes a function of ω. As a result, we have.

$$\lim_{T\to\infty} T a_k = X(\omega) \quad \& \text{ obtain } X(\omega) = \lim_{T\to\infty} T a_k$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \quad — ④$$

In Eq ④, $X(\omega)$ is defined as the Fourier transform of the signal $x(t)$. F-T provides the freq-domain representation of the aperiodic signal $x(t)$. It is also known as analysis equation.

If we multiply & divide the RHS of Eq ① by T & take $T = \frac{2\pi}{\omega_0}$, we get

$$x(t) = \sum_{k=-\infty}^{\infty} T a_k e^{jk\omega_0 t} \cdot \frac{1}{T} = \sum_{k=-\infty}^{\infty} T a_k e^{jk\omega_0 t} \cdot \frac{\omega_0}{2\pi}$$

Now, as $T \to \infty$, $T a_k$ becomes $X(\omega)$. Also, as $T \to \infty$ & $k\omega_0 \to \omega$, the summation becomes an integration from $-\infty$ to ∞. with $\omega_0 \to d\omega \to 0$, we get.

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \quad — ⑤$$

Eq ⑤ is the inverse fourier Transform, it allows us to obtain $x(t)$ from the spectrum $X(\omega)$. It is also known as synthesis equation.

Symbolically, $x(t) \underset{IFT}{\overset{F.T}{\Longleftrightarrow}} X(\omega)$

Note:- $X(\omega) = X(j\omega) = X(j\Omega)$ all are same but representation is different.

In general $X(\omega)$ is a complex valued function of ω. Therefore, we can write $X(\omega)$ is

$$X(\omega) = \underset{\text{real part}}{X_R(\omega)} + j \underset{\text{Imaginary part}}{X_I(\omega)}$$

The plot $|X(\omega)|$ Vs ω is known as Amplitude spectrum & $\angle X(\omega)$ Vs ω is phase spectrum.

The magnitude of $X(\omega)$ is $\quad X(\omega) = \sqrt{[X_R(\omega)]^2 + [X_I(\omega)]^2}$

& the phase of $X(\omega)$ is $\quad \angle X(\omega) = \tan^{-1} \frac{X_I(\omega)}{X_R(\omega)}$

Now,

Fourier Transform of a signal $x(t)$ is

$$\boxed{X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt}$$

&

Inverse Fourier Transform

$$\boxed{x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega}$$

Sufficient conditions for the existence of Fourier transform are similar to those given for the Fourier series. They are Dirichlet conditions :-

1) on any finite interval
   a) $x(t)$ is bounded
   b) $x(t)$ has a finite number of maxima & minima
   c) $x(t)$ has a finite number of discontinuities.

2) $x(t)$ is absolutely integrable. i.e

$$\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$$

Note:- ① These are sufficient condition and not necessary conditions.

② For ex:- Signals $u(t)$, $r(t)$ and $\cos \omega_0 t$ are not absolutely integrable but still possess a Fourier transform.

Properties of Fourier Transform

① Linearity :-

If $x(t) \xrightleftharpoons{F.T} X(\omega)$

& $y(t) \xrightleftharpoons{F.T} Y(\omega)$

then, $z(t) = a x(t) + b y(t) \xrightleftharpoons{F.T} a X(\omega) + b Y(\omega)$

proof :- $Z(\omega) \triangleq \int_{-\infty}^{\infty} z(t) e^{-j\omega t} dt$

$$= \int_{-\infty}^{\infty} [a x(t) + b y(t)] e^{-j\omega t} dt$$

$$Z(\omega) = a \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt + b \int_{-\infty}^{\infty} y(t) e^{-j\omega t} dt$$

$$\boxed{Z(\omega) = a X(\omega) + b Y(\omega)}$$

② Time shift :-

If $x(t) \xrightleftharpoons{F.T} X(\omega)$

then, $y(t) = x(t - t_0) \xrightleftharpoons{F.T} Y(\omega) = e^{-j\omega t_0} X(\omega)$

proof :- $Y(\omega) \triangleq \int_{-\infty}^{\infty} y(t) e^{-j\omega t} dt$

$$= \int_{-\infty}^{\infty} x(t - t_0) e^{-j\omega t} dt$$

put $t - t_0 = \lambda$ & then $dt = d\lambda$

therefore, $Y(\omega) = \int_{-\infty}^{\infty} x(\lambda)\, e^{-j\omega(t_0+\lambda)}\, d\lambda$

$= e^{-j\omega t_0} \int_{-\infty}^{\infty} x(\lambda)\, e^{-j\omega\lambda}\, d\lambda$

Let $\lambda \rightarrow t$, $d\lambda = dt$

$\Rightarrow Y(\omega) = e^{-j\omega t_0} \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$

$$\boxed{Y(\omega) = e^{-j\omega t_0} X(\omega)}$$

(3) Frequency shift :-

If $x(t) \xrightleftharpoons{F.T} X(\omega)$

then, $y(t) \xrightleftharpoons{F.T} e^{j\alpha t} x(t) \xrightleftharpoons{F.T} Y(\omega) = X(\omega - \alpha)$

Proof :- $Y(\omega) \triangleq \int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\, dt$

$= \int_{-\infty}^{\infty} e^{j\alpha t} x(t)\, e^{-j\omega t}\, dt$

$= \int_{-\infty}^{\infty} x(t)\, e^{-j(\omega-\alpha)t}\, dt$

$$\boxed{Y(\omega) = X(\omega - \alpha)}$$

(4) scaling :-

If $x(t) \xrightleftharpoons{F.T} X(\omega)$

then, $y(t) = x(\beta t) \xrightleftharpoons{F.T} Y(\omega) = \frac{1}{|\beta|} X\left(\frac{\omega}{\beta}\right)$

proof :-

$$Y(\omega) \triangleq \int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x(\beta t)\, e^{-j\omega t}\, dt$$

at $\beta > 0$ put $\beta t = \lambda$

then, $dt = \frac{d\lambda}{\beta}$

$$Y(\omega) = \int_{-\infty}^{\infty} x(\lambda)\, e^{-j\omega \frac{\lambda}{\beta}}\, \frac{d\lambda}{\beta}$$

$$= \frac{1}{\beta} \int_{-\infty}^{\infty} x(\lambda)\, e^{-j\omega\left(\frac{\lambda}{\beta}\right)}\, d\lambda$$

$$= \frac{1}{\beta} \int_{-\infty}^{\infty} x(\lambda)\, e^{-j\left(\frac{\omega}{\beta}\right)\lambda}\, d\lambda$$

at $\lambda \to t$, $d\lambda = dt$

$$Y(\omega) = \frac{1}{\beta} \int_{-\infty}^{\infty} x(t)\, e^{-j\left(\frac{\omega}{\lambda}\right)t}\, dt$$

$$\boxed{Y(\omega) = \frac{1}{\beta} X\left(\frac{\omega}{\beta}\right)}$$

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(5) Frequency differentiation:

If $x(t) \overset{F.T}{\rightleftharpoons} X(\omega)$

then $-jt\,x(t) \overset{F.T}{\rightleftharpoons} \dfrac{d}{d\omega} X(\omega)$

Proof:-

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

Differentiating w.r.t ω, we get

$$\frac{dX(\omega)}{d\omega} = \int_{-\infty}^{\infty} x(t)\left[-jt\, e^{-j\omega t}\right] dt$$

$$\frac{dX(\omega)}{d\omega} = -jt \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$\frac{dX(\omega)}{d\omega} = \cancel{-jt} \int_{-\infty}^{\infty} \left[-jt\, x(t)\right] e^{-j\omega t}\, dt$$

$$\frac{dX(\omega)}{d\omega} = -jt\, x(t)$$

$$\boxed{-jt\, x(t) \rightleftharpoons \frac{dX(\omega)}{d\omega}}$$

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(6) Time differentiation :-

If $x(t) \xrightleftharpoons{F.T} X(\omega)$

then $\frac{dx(t)}{dt} \xrightleftharpoons{F.T} j\omega X(\omega)$

Proof :- Inverse F.T

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega \quad —(1)$$

Differentiating both sides w.r.t t, we get

$$\frac{dx(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\,(j\omega e^{j\omega t})\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} [j\omega X(\omega)]\, e^{j\omega t}\, d\omega \quad —(2)$$

Compare (1) & (2)

$$\boxed{\frac{dx(t)}{dt} = j\omega X(\omega)}$$

7 convolution:-

If $x(t) \underset{\longleftarrow}{\xrightarrow{F.T}} X(\omega)$

& $y(t) \underset{\longleftarrow}{\xrightarrow{F.T}} Y(\omega)$

then, $z(t) = x(t) * y(t) \underset{\longleftarrow}{\xrightarrow{F.T}} Z(\omega) = X(\omega)\cdot Y(\omega)$

Proof:-

$$Z(\omega) \triangleq \int_{t=-\infty}^{\infty} z(t)\, e^{-j\omega t}\, dt$$

$$= \int_{t=-\infty}^{\infty} [x(t) * y(t)]\, e^{-j\omega t}\, dt$$

$$= \int_{t=-\infty}^{\infty} \left[\int_{\tau=-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau\right] e^{-j\omega t}\, dt$$

$$= \int_{\tau=-\infty}^{\infty} x(\tau) \left[\int_{t=-\infty}^{\infty} y(t-\tau)\, e^{-j\omega t}\, dt\right] d\tau$$

let $t-\tau = \lambda$, then $dt = d\lambda$

$$Z(\omega) = \int_{\tau=-\infty}^{\infty} x(\tau) \left[\int_{\lambda=-\infty}^{\infty} y(\lambda)\, e^{-j\omega(\tau+\lambda)}\, d\lambda\right] d\tau$$

$$= \int_{\tau=-\infty}^{\infty} x(\tau)\, e^{-j\omega\tau}\, d\tau \cdot \int_{\lambda=-\infty}^{\infty} y(\lambda)\, e^{-j\omega\lambda}\, d\lambda$$

$$\boxed{Z(\omega) = X(\omega)\cdot Y(\omega)}$$

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(8) Integration or Accumulation :-

If $x(t) \xrightleftharpoons{F.T} X(\omega)$

then, $\int_{-\infty}^{t} x(\tau)\,d\tau \xrightleftharpoons{F.T} \frac{1}{j\omega} X(\omega) + \pi X(0)\,\delta(\omega)$

Proof :- $\int_{-\infty}^{t} x(\tau)\,d\tau = x(t) * u(t)$

Hence, $\int_{-\infty}^{t} x(\tau)\,d\tau \xrightleftharpoons{F.T} X(\omega)\,U(\omega)$ —(1)

we know, $u(t) \xrightleftharpoons{F.T} U(\omega) = \pi\,\delta(\omega) + \frac{1}{j\omega}$ [we will see at later stage] —(2)

put eq (2) in eq (1), we get

$\int_{-\infty}^{t} x(\tau)\,d\tau \xrightleftharpoons{F.T} X(\omega)\left[\pi\,\delta(\omega) + \frac{1}{j\omega}\right]$

$\xrightleftharpoons{F.T} \pi X(\omega)\,\delta(\omega) + \frac{1}{j\omega} X(\omega)$

$\cancel{\xrightleftharpoons{F.T} \pi X(0)\,\delta(\omega) + \frac{1}{j\omega} X(\omega)}$

$\left[\because X(\omega)\,\delta(\omega) = X(0)\,\delta(\omega)\right]$

$$\Rightarrow \boxed{\int_{-\infty}^{t} x(\tau)\,d\tau \xrightleftharpoons{F.T} \pi X(0)\,\delta(\omega) + \frac{1}{j\omega} X(\omega)}$$

⑨ Modulation :-

If $x(t) \xrightleftharpoons{F.T} X(\omega)$

& $y(t) \xrightleftharpoons{F.T} Y(\omega)$

then, $z(t) = x(t)y(t) \xrightleftharpoons{F.T} Z(\omega) = \frac{1}{2\pi}\left[X(\omega) * Y(\omega)\right]$

Proof :-

$$Z(\omega) \triangleq \int_{-\infty}^{\infty} z(t) e^{-j\omega t} dt$$

$$= \int_{t=-\infty}^{\infty} x(t) y(t) e^{-j\omega t} dt \quad \text{--- (1)}$$

From Inverse F.T, $x(t) = \frac{1}{2\pi}\int_{\lambda=-\infty}^{\infty} X(\lambda) e^{j\lambda t} d\lambda$

put $x(t)$ in eq (1)

$$Z(\omega) = \int_{t=-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{\lambda=-\infty}^{\infty} X(\lambda) e^{j\lambda t} d\lambda\right] y(t) e^{-j\omega t} dt$$

$$= \frac{1}{2\pi}\int_{\lambda=-\infty}^{\infty} X(\lambda) \int_{t=-\infty}^{\infty} y(t) e^{-j(\omega-\lambda)t} dt \, d\lambda$$

$$= \frac{1}{2\pi}\int_{\lambda=-\infty}^{\infty} X(\lambda) Y(\omega-\lambda) d\lambda$$

$$\boxed{Z(\omega) = \frac{1}{2\pi}\left[X(\omega) * Y(\omega)\right]}$$

(10) Parseval's Theorem or Rayleigh's theorem :-

If $x(t) \xrightleftharpoons{F.T} X(\omega)$

then, $$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

where, $E \rightarrow$ total energy content of the signal $x(t)$

$|X(\omega)|^2 \rightarrow$ Energy density spectrum of the signal $x(t)$

Proof :- $$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$= \int_{t=-\infty}^{\infty} x(t)\, x^*(t)\, dt \quad \text{--- (1)}$$

From Inverse F.T

$$x(t) = \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} X(\omega)\, e^{j\omega t} d\omega$$

taking conjugates on both sides, we get

$$x^*(t) = \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} X^*(\omega)\, e^{-j\omega t} d\omega \quad \text{--- (2)}$$

put eq (2) in eq (1) we get

$$E = \int_{t=-\infty}^{\infty} x(t) \left[\frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} X^*(\omega) e^{-j\omega t} d\omega \right] dt$$

$$E = \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} X^*(\omega) \left[\int_{t=-\infty}^{\infty} x(t) e^{-j\omega t} dt \right] d\omega$$

$$= \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} X^*(\omega) \, X(\omega) \, d\omega$$

$$\boxed{E = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega}$$

Fourier Representation for periodic Discrete Time signals

(Discrete-Time Fourier series)

The Discrete time Fourier series (DTFS) representation of a periodic signal $x[n]$ is

$$x[n] = \sum_{k=\langle N\rangle} X(k)\, e^{jk\Omega_0 n} \quad \rightarrow \text{synthesis eqn.} \quad (1)$$

$$X[k] = \frac{1}{N} \sum_{n=\langle N\rangle} x[n]\, e^{-jk\Omega_0 n} \quad \rightarrow \text{analysis eqn.} \quad (2)$$

Where, $x[n]$ has fundamental period N and fundamental freq. $\Omega_0 = \frac{2\pi}{N}$ radians &

$$\sum_{k=\langle N\rangle} = \sum_{k=0}^{N-1}$$

$X[k] \rightarrow$ Discrete time Fourier series coefficient

→ $$x[n] \xrightleftharpoons{DTFS} X[k]$$

setting $k=0$ in eq (2)

$$X[0] = \frac{1}{N} \sum_{n=\langle N\rangle} x[n]$$

↳ the average value of $x[n]$ over one period

→ The Fourier coefficient are referred as spectral coefficient of $x[n]$.

<u>Convergence of DTFS</u> :- Since, DTFS is a finite series, there are no convergence issues with DTFS.

<u>Periodicity of Fourier Coefficients</u>

$$X[k] = \frac{1}{N} \sum_{n=\langle N \rangle} x(n)\, e^{-jk\Omega_0 n}$$

let $k \rightarrow k+N$, then we have

$$X[k+N] = \frac{1}{N} \sum_{n=\langle N \rangle} x(n)\, e^{-j(k+N)\Omega_0 n}$$

$$= \frac{1}{N} \sum_{n=\langle N \rangle} x[n]\, e^{-jk\Omega_0 n}\, e^{-jN\Omega_0 n}$$

$$= \frac{1}{N} \sum_{n=\langle N \rangle} x(n)\, e^{-jk\Omega_0 n}\, e^{-jN\frac{2\pi}{N} n}$$

$$= \frac{1}{N} \sum_{n=\langle N \rangle} x(n)\, e^{-jk\Omega_0 n}\, e^{-j2\pi n}$$

$\because e^{-j2\pi n} = 1$ for all value of n.

$$\Rightarrow \quad X[k+N] = \frac{1}{N} \sum_{n=\langle N \rangle} x[n]\, e^{-jk\Omega_0 n}$$

$$\boxed{X[k+N] = X[k]}$$

It indicates that the fourier coefficient $X[k]$ are periodic with fundamental period N.

Properties of Discrete Time Fourier Series

① Linearity :-

If $\quad x(n) \xrightleftharpoons{DTFS} X[k]$

& $\quad y[n] \xrightleftharpoons{DTFS} Y[k]$

then,

$$z[n] = a x[n] + b y[n] \xrightleftharpoons{DTFS} Z[k] = a X[k] + b Y[k]$$

In both case, both $x[n]$ & $y[n]$ are assumed to have the same fundamental period $N = \frac{2\pi}{\Omega_0}$

Proof :- $$Z[k] = \frac{1}{N} \sum_{n=\langle N \rangle} z[n] e^{-jk\Omega_0 n}$$

$$= \frac{1}{N} \sum_{n=\langle N\rangle} [a x[n] + b y[n]] e^{-jk\Omega_0 n}$$

$$= a \frac{1}{N} \sum_{n=\langle N\rangle} x[n] e^{-jk\Omega_0 n} + b \frac{1}{N} \sum_{n=\langle N\rangle} y[n] e^{-jk\Omega_0 n}$$

$$\boxed{Z[k] = a X[k] + b Y[k]}$$

② Time shift :-

If $\quad x[n] \xrightleftharpoons{DTFS} X[k]$

then, $\quad y[n] = x[n-n_0] \xrightleftharpoons{DTFS} Y[k] = e^{-jk\Omega_0 n_0} X[k]$

Proof:- $$Y[K] = \frac{1}{N}\sum_{n=\langle N\rangle} y(n) e^{-jk\Omega_0 n}$$

$$= \frac{1}{N}\sum_{n=\langle N\rangle} x[n-n_0]\, e^{-jk\Omega_0 n}$$

~~put n-n0 = m, then~~

$$\cancel{Y[K] = \frac{1}{N}\sum_{n=\langle N\rangle} y(n)\, e^{-jk\Omega_0 n}}$$

$$\cancel{= \frac{1}{N}\sum_{n=\langle N\rangle} x[n-n_0]\, e^{-jk\Omega_0 n}}$$

put $n - n_0 = m$, then

$$Y[K] = \frac{1}{N}\sum_{n=\langle N\rangle} x[m]\, e^{-jk\Omega_0(m+n_0)}$$

$$= e^{-jk\Omega_0 n_0}\,\frac{1}{N}\sum_{m=\langle N\rangle} x[m]\, e^{-jk\Omega_0 m}$$

$$\boxed{Y[K] = e^{-jk\Omega_0 n_0}\, X[K]}$$

③ Frequency shift :-

If $\quad x[n] \xrightleftharpoons{DTFS} X[k]$

then, $\quad y[n] = e^{jk_0\Omega_0 n} x[n] \xrightleftharpoons{DTFS} Y[k] = X[k-k_0]$

Proof :-

$$Y[k] = \frac{1}{N} \sum_{n=\langle N \rangle} y[n] e^{-jk\Omega_0 n}$$

$$= \frac{1}{N} \sum_{n=\langle N \rangle} e^{jk_0\Omega_0 n} x(n) e^{-jk\Omega_0 n}$$

$$= \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-j(k-k_0)\Omega_0 n}$$

$$\boxed{Y[k] = X[k-k_0]}$$

④ Convolution :-

If $\quad x[n] \xrightleftharpoons{DTFS} X[k]$

& $\quad y(n) \xrightleftharpoons{DTFS} Y[k]$

then,

$$z[n] = x[n] \circledast y(n) \xrightleftharpoons{DTFS} N\, X(k)\, Y(k)$$

The sequence $x(n)$, $y(n)$ & $z(n)$ have the same fundamental period equal to N, $\circledast \rightarrow$ denotes periodic convolution.

Proof :- $Z[k] = \frac{1}{N} \sum_{n=\langle N\rangle} z[n]\, e^{-jk\Omega_0 n}$

$$= \frac{1}{N} \sum_{n=\langle N\rangle} \left[x[n] \circledast y[n]\right] e^{-jk\Omega_0 n}$$

$$= \frac{1}{N} \sum_{n=\langle N\rangle} \left[\sum_{l=\langle N\rangle} x[l]\, y[n-l] \right] e^{-jk\Omega_0 n}$$

changing the ~~summation of~~ order of summation

$$Z[k] = \frac{1}{N} \left[\sum_{l=\langle N\rangle} x[l] \sum_{n=\langle N\rangle} y[n-l]\, e^{-jk\Omega_0 n} \right]$$

put $n-l=m$, we get

$$Z[k] = \frac{1}{N} \left[\sum_{l=\langle N\rangle} x[l] \sum_{m=\langle N\rangle} y[m]\, e^{-jk\Omega_0 m} e^{-jk\Omega_0 l} \right]$$

$$= \frac{1}{N} \left[\sum_{l=\langle N\rangle} x[l]\, e^{-jk\Omega_0 l} \sum_{m=\langle N\rangle} y[m]\, e^{-jk\Omega_0 m} \right]$$

$$= \frac{1}{N} \left[N\, X[k] \cdot N\, Y[k] \right]$$

$$\boxed{Z[k] = N\, X[k]\, Y[k]}$$

(5) Modulation or Multiplication :-

If $\quad x(n) \xrightleftharpoons{DTFS} X[k]$

& $\quad y(n) \xrightleftharpoons{DTFS} Y[k]$

then, $\quad Z(n) = x[n]y(n) \xrightleftharpoons{DTFS} Z(k) = X[k] \circledast Y(k)$

It is assumed that $x(n)$, $y[n]$ and $Z(n)$ have the same fundamental period equal to N.

Proof :-

$$Z(k) = \frac{1}{N} \sum_{n=\langle N \rangle} Z[n] \, e^{-jk\Omega_0 n}$$

$$= \frac{1}{N} \sum_{n=\langle N \rangle} x[n] \, y(n) \, e^{-jk\Omega_0 n} \quad \text{--- (1)}$$

the synthesis equation for $x(n)$ is

$$x[n] = \sum_{l=\langle N \rangle} x[l] \, e^{jl\Omega_0 n} \quad \text{--- (2)}$$

put eq (2) in eq (1), we get

$$Z(k) = \frac{1}{N} \sum_{l=\langle N \rangle} X[l] \sum_{n=\langle N \rangle} y(n) \, e^{-j(k-l)\Omega_0 n}$$

$$= \sum_{l=\langle N \rangle} X[l] \, Y[k-l]$$

$$\boxed{Z(k) = X[k] \circledast Y(k)}$$

(6) Parseval's theorem :-

If $x[n] \xrightleftharpoons{DTFS} X[k]$

then, $$P = \frac{1}{N}\sum_{n=\langle N\rangle} |x[n]|^2 = \sum_{k=\langle N\rangle} |X[k]|^2$$

where, P is the average power of the periodic sequence $x(n)$.

Proof :- $$P = \frac{1}{N}\sum_{n=\langle N\rangle} |x[n]|^2$$

$$= \frac{1}{N}\sum_{n=\langle N\rangle} x[n]\, x^*[n]$$

$$= \frac{1}{N}\left[\sum_{n=\langle N\rangle} x[n] \left(\sum_{k=\langle N\rangle} X^*[k]\, e^{-jk\Omega_0 n}\right)\right]$$

changing the order of summation, we get

$$P = \sum_{k=\langle N\rangle} X^*[k]\left[\frac{1}{N}\sum_{n=\langle N\rangle} x[n]\, e^{-jk\Omega_0 n}\right]$$

$$= \sum_{k=\langle N\rangle} X^*[k]\, X[k]$$

$$= \sum_{k=\langle N\rangle} |X[k]|^2$$

$$\Rightarrow P = \frac{1}{N}\sum_{n=\langle N\rangle} |x[n]|^2 = \sum_{k=\langle N\rangle} |X[k]|^2$$

$|X[k]|^2 \rightarrow$ Distribution of power as a function of freq. k & is defined as the power density spectrum of $x[n]$.

(7) Duality:-

If $\quad x[n] \overset{DTFS}{\rightleftharpoons} X[k]$

then $\quad X[n] \overset{DTFS}{\rightleftharpoons} \frac{1}{N} x[-k]$

proof:-

$$x[n] = \sum_{k=\langle N \rangle} X[k] e^{jk\Omega_0 n}$$

Replacing n by $-n$, we get

$$x[-n] = \sum_{k=\langle N \rangle} X[k] e^{-jk\Omega_0 n}$$

Replacing n by k & k by n we get..

$$x[-k] = \sum_{n=\langle N \rangle} X[n] e^{-jk\Omega_0 n}$$

multiplying both sides by $\frac{1}{N}$, we get

$$\frac{1}{N} x[-k] = \frac{1}{N} \sum_{n=\langle N \rangle} X[n] e^{-jk\Omega_0 n}$$

$$\Rightarrow \boxed{X[n] \overset{DTFS}{\rightleftharpoons} \frac{1}{N} x[-k]}$$

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(8) Symmetry:-

when $x[n]$ is real,

$$X[-k] = X[N-k]$$
$$= X^*[k]$$

where * denotes the complex conjugate

when $x(n)$ is real,

let $x(n) = x_e[n] + x_o[n]$

where, $x_e[n]$ & $x_o[n]$ are the even and odd components of $x(n)$. respectively

let $x(n) \xrightleftharpoons{DTFS} X[k]$

then,

$$x_e[n] \xrightleftharpoons{DTFS} Re\{X[k]\}$$
$$x_o[n] \xrightleftharpoons{DTFS} j\,Im\{X[k]\}$$

Thus, if $x(n)$ is real and even, the its fourier coefficient are real, while if $x(n)$ is real and odd, its fourier coefficient are imaginary